

ON QUATERNION DETERMINANTAL POINT FIELDS

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Abstract

We study a quaternion version of determinantal random point fields defined by using the Moore-Dyson quaternion determinant. First, we give sufficient conditions that ensure that a self-dual quaternion kernel defines a valid random point field, and then we prove a CLT for quaternion determinantal fields. The proofs are based on a quaternion extension of the Cauchy-Binet determinantal identity.

1. INTRODUCTION

A determinantal random point field is a random collection of points on a nice topological space with the probability distribution that can be written as a determinant. The determinantal point fields describe various mathematical objects including eigenvalues of random matrices, zeros of random analytic functions, non-intersecting random paths, and spanning trees on networks. It is conjectured that they are also related to other important objects such as Riemann's zeta function zeros and the spectrum of chaotic dynamical systems. See [24], [13], [12] and [4] for reviews.

The definition of the determinantal point fields uses the standard matrix determinant. However, in some applications, the distribution of a random point field can be represented as a determinant of a quaternion matrix. An important example is provided by eigenvalues of orthogonal and symplectic random matrix ensembles. It is natural to call these fields as quaternion determinantal fields and study their properties. This paper makes some initial steps in this direction.

In order to define a quaternion determinantal point field, we need to establish some preliminary notations.

A random point field $\mathcal{X} = (X, \mathcal{B}, \mathbb{P})$ on the set $\Lambda \subset \mathbb{R}^n$ (or \mathbb{Z}^n) is a probability measure \mathbb{P} on the space X of all possible countable configurations of points in Λ . It is convenient to think about this object as a collection of functions that sends every n -tuple of non-negative integers (k_1, \dots, k_n) and every n -tuple of Borel subsets of Λ , (A_1, \dots, A_n) to a non-negative number, which can be interpreted as a probability to find k_i points in the set A_i . These functions are to

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satisfy some consistency conditions, which we do not specify here. The reader is advised to consult paper [14] or book [6] for more detail.

Let $\#(A_i)$ denote the number of points of \mathcal{X} located in the set A_i .

Definition 1.1. A locally integrable function $R_k: \Lambda^k \rightarrow \mathbb{R}_+^1$ is called a k -point correlation function of a random point field $\mathcal{X} = (X, \mathcal{B}, \mathbb{P})$ with respect to background measure μ , if for any disjoint Borel subsets A_1, \dots, A_m and any non-negative integers k_1, \dots, k_m , such that $\sum_{i=1}^m k_i = k$, the following formula holds:

$$\mathbb{E} \prod_{i=1}^m [\#(A_i) \dots (\#(A_i) - k_i + 1)] = \int_{A_1^{k_1} \times \dots \times A_m^{k_m}} R_k(x_1, \dots, x_k) d\mu(x_1) \dots d\mu(x_k), \quad (1)$$

where \mathbb{E} denote expectation with respect to measure \mathbb{P} .

Note that on the left is the expected number of ordered configurations of points such that set A_i contains k_i points. Note also that the correlation functions are defined only up to sets of measure 0.

Definition 1.2. A random point field \mathcal{X} on the set $\Lambda \subset \mathbb{R}^n$ will be called a quaternion determinantal point field if its correlation functions can be written as quaternion determinants:

$$R_m(x_1, \dots, x_m) = \text{Det}_M(K(x_i, x_j))|_{1 \leq i, j \leq m}, \quad (2)$$

where $K(x, y)$ is a self-dual quaternion kernel (that is, $K(y, x) = (K(x, y))^*$), and Det_M is the Moore-Dyson quaternion determinant.

(We will give more information about quaternions and quaternion determinants in the next section.)

Example 1. It was shown by Dyson [8] that the eigenvalues of the circular orthogonal and symplectic ensembles of random matrices form a quaternion determinantal field. Later this result was extended by Mehta (see Chapters 7 and 8 in [18]) to the case of Gaussian orthogonal and symplectic ensembles. We will say more about the circular symplectic ensemble in the last section.

Example 2. (Ginibre on \mathbb{Q}) Let z and w be in \mathbb{Q} , where \mathbb{Q} denotes quaternions over the real field, and define the kernel

$$K_n(z, w) = \sum_{k=0}^n \frac{z^k (w^*)^k}{(k+1)!}. \quad (3)$$

This kernel defines a random field on the set of all real quaternions $\mathbb{Q} \simeq \mathbb{R}^4$ taken with the background measure $d\mu(z) = \pi^{-2} e^{-|z|^2} dm(z)$, where $dm(z)$ is the Lebesgue measure on \mathbb{Q} . The fact that kernel (3) defines a random point field follows from Proposition 4.3 below.

Recall for comparison that the usual Ginibre random point field on \mathbb{C} gives a distribution of eigenvalues of a random n -by- n complex Gaussian matrix. It is determinantal with the

kernel

$$K_n(z, w) = \sum_{k=0}^n \frac{z^k (\overline{w})^k}{k!},$$

where z and w are in \mathbb{C} and the background measure is $d\mu(z) = \pi^{-1} e^{-|z|^2} dm(z)$. (See [11], Section 15.1 in [18], and Section 4.3.7 in [12].)

Example 3. (quaternion Ginibre on \mathbb{C}) This is another generalization of the Ginibre random point field. See Section 15.2 in [18] for details. Let

$$\phi_N(u, v) = \frac{1}{2\pi} \sum_{0 \leq i \leq j \leq N-1} \frac{2^j j!}{2^i i!} \frac{1}{(2j+1)!} (u^{2i} v^{2j+1} - v^{2i} u^{2j+1}),$$

and define the quaternion kernel by its complex matrix representation:

$$\varphi(K_N(z, w)) = \begin{pmatrix} \phi_N(w, \overline{z}) & \phi_N(\overline{w}, \overline{z}) \\ \phi_N(z, w) & \phi_N(z, \overline{w}) \end{pmatrix}.$$

(The definition of the bijective map $\varphi: \mathbb{Q}_{\mathbb{C}} \rightarrow M_2(\mathbb{C})$ is standard and given in the next section.) Then this kernel defines the determinantal kernel with respect to the signed background measure $d\mu(z) = e^{-|z|^2} (z - \overline{z}) dm(z)$.

Example 4 (Bergman kernel on \mathbb{Q}). Let z and w be in \mathbb{Q} , and define the kernel

$$K_n(z, w) = \sum_{k=0}^n (k+2) z^k (w^*)^k. \quad (4)$$

This kernel defines a random field on the unit disc of real quaternions with the background measure $d\mu(z) = \pi^{-2} dm(z)$, where $dm(z)$ is the Lebesgue measure.

For comparison, the Bergman kernel on the unit disc in \mathbb{C} is given by

$$K_n(z, w) = \sum_{k=0}^n (k+1) z^k (\overline{w})^k,$$

and corresponds to the distribution of zeros of power series with i.i.d complex Gaussian coefficients. (See Section 15.2 in [12].)

The quaternion determinantal point fields are a particular case of the Pfaffian point fields which were studied in [21], [25] and [3].

Which self-dual quaternion kernels define random point fields?

Note that a correlation function is automatically symmetric if it is defined as in (2). That is, we have

$$R_m(x_{\sigma(1)}, \dots, x_{\sigma(m)}) = R_m(x_1, \dots, x_m)$$

for every permutation $\sigma \in S_m$. Hence, by the Lenard criterion ([15] and [16]), positivity is a necessary and sufficient condition that ensures that a kernel corresponds to a random point field. This condition can be explained as follows. Let X be the space of configurations of points in Λ . Let $\varphi = \{\varphi_k\}$ denote an arbitrary sequence of real-valued functions φ_k over Λ^k which have the property that they are zero if at least one of its arguments is outside of a

compact set K in Λ . Define operator S on φ as follows: S maps sequence φ to a function $S\varphi$ on X

$$(S\varphi)(x) = \sum \varphi_k(x_{i_1}, \dots, x_{i_k})$$

where $x = (x_1, x_2, \dots)$ is an enumeration of a point configuration in X , and the sum is extended over all finite sequences of distinct positive integers (i_1, \dots, i_k) including the empty sequence. (This sum is essentially finite since the definition of the space of configurations requires that the number of points of any configuration in any bounded subset of Λ be finite.) The positivity condition says that if $(S\varphi)(x) \geq 0$ for all $x \in X$ (including the empty sequence), then it must be true that

$$\int \varphi(x) d\rho := \varphi_0 + \sum_{k=1}^N \int_{\Lambda^k} \varphi_k(x_1, \dots, x_k) R_k(x_1, \dots, x_k) d\mu(x_1) \dots d\mu(x_k) \geq 0.$$

However, this criterion is often difficult to verify.

In the case of the usual determinantal fields with self-adjoint kernels a much simpler criterion is given by the Macchi-Soshnikov theorem (see [17] and [24]) that says that a self-adjoint kernel $K(x, y)$ defines a determinantal point field if and only if the corresponding operator K is in the trace class and all its eigenvalues are in the interval $[0, 1]$. This is equivalent to the requirement that both K and $I - K$ are non-negative definite.

Our first result is a weaker version of this theorem for quaternion determinantal fields.

Suppose that a quaternion kernel K can be written as follows:

$$K(x, y) = \sum_{k=1}^{\infty} \lambda_k u_k(x) u_k^*(y), \quad (5)$$

where λ_k are scalar and $u_k(x)$ is an orthonormal system of (possibly complex) quaternion functions:

$$\int_{\Lambda} u_k^*(x) u_l(x) d\mu(x) = \delta_{kl}.$$

(The series in (5) are assumed to be absolutely convergent almost everywhere.) We will say in this case that $K(x, y)$ has a *diagonal form* with eigenvalues λ_k . If λ_k are real and $u_k(x)$ is an orthonormal system of real quaternion functions, then we will say that $K(x, y)$ has a *real diagonal form*.

Theorem 1.3. *Suppose that a quaternion kernel $K(x, y)$ has a real diagonal form with eigenvalues λ_k . Assume that all $\lambda_k \in [0, 1]$, and that $\sum_{k=1}^{\infty} \lambda_k < \infty$. Then the kernel $K(x, y)$ defines a determinantal point field with finite expected number of points.*

The conditions of this theorem are sufficient but not necessary since there exist self-dual quaternion kernels which do not have a real diagonal form and still define a valid point field. For example, let space Λ consist of two points and has the counting background measure. We can define a quaternion determinant point field on this space by setting the kernel equal to a

2-by-2 quaternion matrix

$$K = \frac{1}{2} \begin{pmatrix} 1 & -a \\ a & 1 \end{pmatrix},$$

where $a = (3i - 5j)/4$ so that $a^2 = -1$. This matrix is the matrix of the orthogonal projection on vector $(1, a)$. Clearly, this matrix is self-dual with determinant zero. Hence, it defines a random field that has exactly 1 point uniformly distributed on Λ . On the other hand, this kernel has a diagonal form with real eigenvalues but it does not have a real diagonal form.

Moreover, there exist self-dual quaternion kernels which do not have any diagonal form at all and still define a valid point field. Here is an example of this situation. Again, let the space Λ consist of two points, and consider the kernel equal to a 2-by-2 quaternion matrix

$$K = \begin{pmatrix} 1 & a \\ -a & 1 \end{pmatrix},$$

where $a = i\mathbf{i} - \mathbf{j}$, so that $a^2 = 0$. Clearly, this matrix is self-dual with determinant 1. Hence, it defines a random field with $R_1 = 1$ and the correlation matrix

$$R_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This random field has exactly 2 points. On the other hand, K does not have a diagonal form because if it had such a form then it would be a projection matrix, but an explicit calculation shows that $K^2 \neq K$.

(Note by the way that the determinantal field defined by K is the same as the field defined by the identity matrix. Hence, two different quaternion kernels can correspond to the same random point field.)

On the other hand, in Theorem 1.3, it is not possible to omit the requirement that the eigenfunctions $u_k(x)$ are real quaternion. Indeed, there are kernels that have a diagonal form with real eigenvalues in the interval $[0, 1]$ but that do not define a random point field. Consider for example, the same two-point space Λ and the kernel

$$K = \frac{4}{3} \begin{pmatrix} 1 & i/2 \\ i/2 & -1/4 \end{pmatrix}.$$

It has a diagonal form with $\lambda = 1$ and $u = (2/\sqrt{3})[1, i/2]^*$ but it does not define a random field since the first correlation function is negative at the second point.

Most of the examples from random matrix theory are concerned with the kernels that do not have a real diagonal form. Orthogonal random matrix ensembles correspond to kernels without diagonal form and symplectic ensembles correspond to kernels without real diagonal form. Hence, it appears desirable to improve Theorem 1.3. What are sufficient and necessary conditions for a self-dual quaternion kernel to define a determinantal random point field?

While this question is not answered in this paper, we can slightly generalize Theorem 1.3 to give sufficient conditions for kernels without a real diagonal form. We will say a kernel

$K(x, y)$ has a *quasi-real diagonal form* (or simply call it quasi-real) if it has a diagonal form with real eigenvalues. We will call it *positive* if

$$\text{Det}_M (K(x_i, x_j))|_{1 \leq i, j \leq m} \geq 0$$

for all $m \in \mathbb{Z}^+$ and all x_1, \dots, x_m . It is remarkable that quasi-real kernels with positive eigenvalues are not necessarily positive (as the example above shows).

We will call a quasi-real kernel $K(x, y)$ *completely positive* if every of the kernels

$$K_I = \sum_{i \in I} u_i(x) u_i^*(y)$$

is positive, where $\{u_i(x)\}$ is an orthonormal set of eigenfunctions of $K(x, y)$ and $I = (i_1, \dots, i_m)$ denote an ordered subset of indices of all eigenfunctions.

Theorem 1.4. *Suppose that a quaternion kernel $K(x, y)$ is finite-rank, quasi-real and completely positive. Assume that all $\lambda_k \in [0, 1]$. Then the kernel $K(x, y)$ defines a determinantal point field.*

Remark: We omit the assumption that $\sum_{k=1}^{\infty} \lambda_k < \infty$ which we imposed in Theorem 1.3 since we assume that the kernel is finite-rank. It should be possible to extend this theorem to a more general case when there are infinite number of λ_k and $\sum_{k=1}^{\infty} \lambda_k < \infty$.

Still, even if we restrict attention to kernels with a diagonal form, the conditions of Theorem 1.4 are not necessary. For example, consider the two-point space Λ with the counting measure μ . Let

$$a = (1 + 2i) + \left(\frac{19}{10} - \frac{20}{19}i \right) \mathbf{i},$$

and define the kernel by the matrix

$$K = \frac{1}{2} \begin{pmatrix} 1 & a \\ a^* & 1 \end{pmatrix}.$$

Then $\text{Det}_M(K) \approx 0.3745$ and therefore the kernel defines a valid random point field on Λ . On the other hand the eigenvalues are $\lambda_{1,2} \approx \frac{1}{2} \pm 0.3529i$ and therefore the kernel is not quasi-real.

More generally, we have the following example.

Let λ be a complex number and assume that $|\lambda| \leq 1$, and $\text{Re} \lambda \geq |\lambda|^2$. Let $u(x)$ and $v(x)$ be two orthonormal quaternion functions on Λ and let $K(x, y) = u(x) u^*(y) + v(x) v^*(y)$. Assume that $u(x) u^*(x) = v(x) v^*(x)$ for every x and that it is true that at

$$\text{Det}_M \begin{pmatrix} K(x, x) & K(x, y) \\ K(y, x) & K(y, y) \end{pmatrix} \geq 0$$

for all $x, y \in \Lambda$.

Then the kernel

$$\lambda u(x) u^*(y) + \bar{\lambda} v(x) v^*(y)$$

corresponds to a quaternion determinantal random field on Λ . Indeed, it is a mix of the fields with kernels $K(x, y)$, $u(x)u^*(y)$, and 0, taken with probabilities $p_2 = |\lambda|^2$, $p_1 = 2(\operatorname{Re}\lambda - |\lambda|^2)$ and $p_0 = 1 - p_1 - p_2$.

Numerical data suggest that in the case of the symplectic random matrix ensembles the kernel is quasireal.

The proof of Theorem 1.3 is based on the following result.

Suppose $K(x, y)$ has a real diagonal form with eigenvalues λ_k , that all $\lambda_k \in [0, 1]$, and that $\sum_{k=1}^{\infty} \lambda_k < \infty$. Define a random kernel

$$K_{\xi}(x, y) = \sum_{k=1}^{\infty} \xi_k u_k(x) u_k^*(y), \quad (6)$$

where ξ_k are independent Bernoulli random variables. The random variable ξ_k takes value 1 with probability λ_k .

We will prove later that this kernel defines a quaternion determinantal field.

Theorem 1.5. *Suppose $K(x, y)$ has a real diagonal form with eigenvalues λ_k , that all $\lambda_k \in [0, 1]$, and that $\sum_{k=1}^{\infty} \lambda_k < \infty$. Let \mathcal{X}_{ξ} be the random point process which is a mix of the quaternionic determinantal processes with random kernels $K_{\xi}(x, y)$ defined in (5). Then, \mathcal{X}_{ξ} is the quaternion determinantal process with the kernel $K(x, y)$.*

This theorem is a quaternion version of Theorem 7 in [13]. In order to prove Theorem 1.5, we need an analogue of the Cauchy-Binet formula for quaternion determinants which we state in Section 3 and prove in Appendix.

An analogous result holds for the quasi-real case.

Theorem 1.6. *Suppose $K(x, y)$ is finite-rank, quasi-real and completely positive with eigenvalues $\lambda_k \in [0, 1]$. Let \mathcal{X}_{ξ} be the random point process which is a mix of the quaternionic determinantal processes with random kernels $K_{\xi}(x, y)$ defined in (6). Then, \mathcal{X}_{ξ} is the quaternion determinantal process with the kernel $K(x, y)$.*

Another consequence of Theorem 1.5 is the central limit theorem for the number of points in quaternion determinantal fields.

Theorem 1.7. *Let \mathcal{X}_n be a sequence of quaternion determinantal point fields defined on a set $D \subseteq \mathbb{R}$ with kernels that have a real diagonal form with eigenvalues in the interval $[0, 1]$. Let \mathcal{N}_n denote the number of points of \mathcal{X}_n in D . Suppose that every \mathcal{N}_n is finite with probability 1, and that $\operatorname{Var}(\mathcal{N}_n) \rightarrow \infty$ as $n \rightarrow \infty$. Then, the sequence of random variables*

$$\frac{\mathcal{N}_n - \mathbb{E}(\mathcal{N}_n)}{\sqrt{\operatorname{Var}(\mathcal{N}_n)}}$$

approaches a standard Gaussian random variable in distribution.

This is a quaternion analog of a theorem that was proved by Costin and Lebowitz in [5] and Diaconis and Evans in [7] for particular cases, and by Soshnikov in [26] for general

determinantal ensembles. Later, a simplified proof was suggested in [13] and we use its main idea to prove our theorem.

For the quasi-real case we have the following parallel result.

Theorem 1.8. *Let \mathcal{X}_n be a sequence of quaternion determinantal point fields defined on a set $D \subseteq \mathbb{R}$ with kernels that finite-rank, quasi-real and completely positive, and which have eigenvalues in the interval $[0, 1]$. Let \mathcal{N}_n denote the number of points of \mathcal{X}_n in D . Suppose that every \mathcal{N}_n is finite with probability 1, and that $\text{Var}(\mathcal{N}_n) \rightarrow \infty$ as $n \rightarrow \infty$. Then, the sequence of random variables*

$$\frac{\mathcal{N}_n - \mathbb{E}(\mathcal{N}_n)}{\sqrt{\text{Var}(\mathcal{N}_n)}}$$

approaches a standard Gaussian random variable in distribution.

We will see in the final section that the kernel of the circular symplectic ensemble of random matrices is quasi-real, which gives an independent proof that it defines a determinantal point field on the unit circle. Numerical evaluations suggest that this kernel remains quasi-real even if it is restricted to an interval of the unit circle. However, the proof of this claim is elusive.

The rest of the paper is organized as follows. Section 2 provides some background on quaternion matrices and determinants. Section 3 formulates the Cauchy-Binet identity. Section 4 proves Theorem 1.3, Section 5 proves Theorem 1.7, and Section 6 provides an illustration of the general theorems by using the circular symplectic ensemble of random matrices. The appendix contains a proof of the quaternion Cauchy-Binet identity.

2. QUATERNION MATRICES AND DETERMINANTS

Recall that the algebra of real quaternions \mathbb{Q} is a non-commutative division algebra isomorphic to a subalgebra of the algebra of all 2-by-2 complex matrices $M_2(\mathbb{C})$ which is generated over \mathbb{R} by the identity matrix and matrices

$$\mathbf{i} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \mathbf{j} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \text{ and } \mathbf{k} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

which are called quaternion units. The quaternions are typically written as $q = s + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, where s, x, y, z are real. The conjugate of q is $q^* = s - x\mathbf{i} - y\mathbf{j} - z\mathbf{k}$, and the norm of q is $|q| := (q^*q)^{1/2} = (s^2 + x^2 + y^2 + z^2)^{1/2}$.

It is also useful to define complex quaternions as linear combinations of the identity element and quaternion units \mathbf{i}, \mathbf{j} , and \mathbf{k} with complex coefficients. The algebra of complex quaternions $\mathbb{Q}_{\mathbb{C}}$ is isomorphic to $M_2(\mathbb{C})$. In the case of complex quaternions, we use the same definition for the conjugate, $q^* = s - x\mathbf{i} - y\mathbf{j} - z\mathbf{k}$, and we define the norm as $\|q\| := \left(|s|^2 + |x|^2 + |y|^2 + |z|^2 \right)^{1/2}$. We say real quaternions for quaternions with real coefficients.

In terms of 2-by-2 matrices, if

$$q = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then its conjugate is

$$q^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Quaternion matrices are matrices whose entries are quaternions. They obey the usual rules of matrix addition and multiplication. The *dual* of a quaternion matrix X is defined as a matrix X^* , for which $(X^*)_{lk} = (X_{kl})^*$. *Self-dual* quaternion matrices are defined by the property that $X^* = X$. *Unitary* quaternion matrices are matrices with the property that $X^*X = XX^* = I$, where I is the identity matrix.

If we represent each quaternion by a 2-by-2 complex matrix, then matrix X becomes a $2n$ -by- $2n$ complex matrix which we denote $\varphi(X)$. Let J be a $2n$ -by- $2n$ block-diagonal matrix with the blocks

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

on the main diagonal. Then it is easy to check that

$$\varphi(X^*) = -J\varphi(X)^T J.$$

Complex matrices A that satisfy the condition $A^T = -JAJ$ are called *skew-Hamiltonian*, and the matrices V that satisfy the condition $V^T J V = J$ are called *symplectic*. Hence, a quaternion matrix X is self-dual if and only if $\varphi(X)$ is skew-Hamiltonian, and X is unitary if and only if $\varphi(X)$ is *symplectic*.

Some of more involved linear algebra concepts are known to be valid for quaternion matrices with real quaternion entries, for example, matrix eigenvalues and singular value decomposition. A number λ is called an eigenvalue of quaternion matrix X if for some non-zero quaternion vector v , we have $Xv = v\lambda$. (These are the right eigenvalues of X , which are the most convenient in applications.) It is easy to see that if λ is an eigenvalue, then $q^{-1}\lambda q$ is also an eigenvalue for any quaternion q . However, for self-dual quaternion matrices with real quaternion entries, all eigenvalues are real and it is possible to show that any n -by- n matrix X of this type has exactly n eigenvalues (counting with multiplicities); see Zhang [28].

Furthermore, it is possible and useful to generalize the concept of determinant to quaternionic matrices. There are several sensible ways to do this and in this paper we will only use the Moore-Dyson and Study determinants. Interested reader can find details in a review paper by Aslaksen [2].

The Study determinant of X is defined as the usual determinant of the complex representation of X ,

$$\text{Det}_S(X) := \det(\varphi(X)).$$

It can also be defined in a slightly different way for real quaternionic matrices. If

$$X = X_1 + X_2\mathbf{i} + X_3\mathbf{j} + X_4\mathbf{k},$$

where X_1, X_2, X_3 , and X_4 are real, then we define two complex matrices $A = X_1 + X_2i$ and $B = X_3 + X_4i$. Then, $\psi(X)$ is a $2n$ -by- $2n$ complex matrix defined by the following rule:

$$\psi(X) = \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix}, \quad (7)$$

where \bar{A} is the conjugate of matrix A , that is, $(\bar{A})_{kl} = (A_{kl})^*$, and similarly for \bar{B} . The matrix $\psi(X)$ is called the *complex adjoint* of matrix X . Then

$$\text{Det}_S(X) = \det(\psi(X)),$$

where the determinant on the right is the usual determinant of complex matrices.

The Study determinant is multiplicative: $\text{Det}_S(AB) = \text{Det}_S(A) \text{Det}_S(B)$ for square matrices A and B .

The Moore-Dyson determinant of a self-dual matrix X with real quaternion entries can be defined as the product of the right eigenvalues of the matrix. Remarkably, this determinant can also be extended to all matrices by using a variant of the Cayley combinatorial formula for the determinant. Namely, let S_n be the group of permutations of the set $\{1, \dots, n\}$. Write every permutation σ as a product of cycles:

$$\sigma = (n_1 i_2 \dots i_s) (n_2 j_2 \dots j_t) \dots (n_r k_2 \dots k_l),$$

where n_i are the largest elements of each cycle and $n_1 > n_2 > \dots > n_r$. Then we can write

$$\text{Det}_M(X) = \sum_{\sigma} \varepsilon(\sigma) (X_{n_1 i_2} X_{i_2 i_3} \dots X_{i_s n_1}) \dots (X_{n_r k_2} X_{k_2 k_3} \dots X_{k_l n_r}),$$

where $\varepsilon(\sigma) = (-1)^{n-r}$ is the sign of the permutation σ . (see [19] and [8]).

Note that this definition allows one to calculate the quantity $\text{Det}_M(X)$ for an arbitrary quaternionic matrix. Dyson established that this quantity is scalar for a self-dual matrix, that is, in this case, the \mathbf{i}, \mathbf{j} , and \mathbf{k} components of the determinant are zero.

The Moore-Dyson quaternion determinant of a self-dual quaternion matrix can also be written as the Pfaffian of a related complex matrix. If X is a self-dual quaternion matrix, then $-J\varphi(X)$ is antisymmetric (that is, $[-J\varphi(X)]^T = J\varphi(X)$), and we can compute the Pfaffian of this matrix. We have

$$\text{Det}_M(X) = \text{Pf}(-J\varphi(X)).$$

(see [8] and Proposition 6.1.5 on p. 238 in Forrester's book [10]).

In terms of the transformation ψ , this can be written as follows. Let X be a real quaternion matrix and let A and B be defined as in (7). Let

$$\tilde{J} = \psi(\mathbf{j}) = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

Then,

$$-\tilde{J}\psi(X) = \begin{pmatrix} \overline{B} & \overline{A} \\ -A & B \end{pmatrix}.$$

If the real quaternion matrix X is self-dual, then $-\tilde{J}\psi(X)$ is antisymmetric, and it can be shown that

$$\text{Det}_M(X) = -\text{Pf}\left(-\tilde{J}\psi(X)\right). \quad (8)$$

The Study and Moore-Dyson determinants are related by the following formula:

$$\text{Det}_S(X) = \text{Det}_M(X^*X). \quad (9)$$

(See formula (6.13) on page 239 in [10] or Corollary 5.1.3 on p. 75 in [18].)

3. CAUCHY-BINET FORMULA

Determinants of matrices with quaternion entries have been studied for almost a hundred years. (Study wrote a paper [27] about them in 1920, and Moore presented his definition [19] in 1922). In the 1970s, this subject got a boost after Dyson re-discovered Moore's determinant and related it to the distribution of eigenvalues of random matrices, [8]. In recent years, the entire subject of quaternion linear algebra received an increased share of researchers' attention because of its applications in image processing ([20], [9], [22], [1]).

Unfortunately, while there are several variants of quaternion determinants, none of them enjoys all the properties of the usual determinant. The validity of each standard determinantal identity has to be checked individually.

Recall that the Cauchy-Binet identity states that if A is an m -by- n matrix and B is an n -by- m matrix, with $n \geq m$, then

$$\det(AB) = \sum_I \det(A^I) \det(B^I),$$

where the summation is over $I = (i_1 < i_2 < \dots < i_m)$, the ordered subsets of $\{1, \dots, n\}$ that consist of m elements. Matrices A^I and B^I are square matrices that consist of m columns of A and m rows of B , respectively, with indices in I .

An implicit assumption in this result is that the entries of the matrices A and B are from a commutative ring, for example from a field of complex numbers. Unfortunately, in this form the Cauchy-Binet identity fails for the quaternion determinants.

It turns out that a weaker form of the Cauchy-Binet identity still holds for quaternion determinants.

Theorem 3.1. *Let C be an n -by- m matrix with (possibly complex) quaternion entries, $n \geq m$, and let C^* be the dual of C . Then*

$$\text{Det}_M(C^*C) = \sum_I \text{Det}_M\left((C^I)^* C^I\right),$$

where the summation is over $I = (i_1 < i_2 < \dots < i_m)$, the \geq ordered subsets of $\{1, \dots, n\}$ that consist of m elements, and where C^I consists of rows i_1, \dots, i_m of C .

A proof of this theorem is in Appendix.

Corollary 3.2. *Suppose that Λ is an n -by- n diagonal matrix with scalar entries λ_i on the main diagonal and that C is an n -by- m quaternion matrix, $n \geq m$. Then, we have.*

$$\text{Det}_M (C^* \Lambda C) = \sum_{\substack{I=(i_1, \dots, i_m) \\ i_1 < \dots < i_m}} \lambda_{i_1} \dots \lambda_{i_m} \text{Det}_M \left((C^I)^* C^I \right).$$

Proof of Corollary 3.2: Let $\Lambda^{1/2}$ be an n -by- n diagonal matrix with scalar entries such that $(\Lambda^{1/2})^2 = \Lambda$. For $I = (i_1, \dots, i_m)$, let $(\Lambda^{1/2})^{II}$ denote an m -by- m matrix which is formed by taking entries at the intersection of rows and columns i_1, \dots, i_m in matrix $\Lambda^{1/2}$. We write

$$\begin{aligned} \text{Det}_M (C^* \Lambda C) &= \sum_I \text{Det}_M \left((\Lambda^{1/2} C)^{I*} (\Lambda^{1/2} C)^I \right) \\ &= \sum_I \text{Det}_S \left((\Lambda^{1/2} C)^I \right) \\ &= \sum_I \text{Det}_S \left((\Lambda^{1/2})^{II} C^I \right) \\ &= \sum_I \text{Det}_S \left((\Lambda^{1/2})^{II} \right) \text{Det}_S (C^I) \\ &= \sum_{I=(i_1, \dots, i_m)} \lambda_{i_1} \dots \lambda_{i_m} \text{Det}_M \left((C^I)^* C^I \right). \end{aligned}$$

The first line is the Cauchy-Binet identity. The second line follows from (9). The fourth line follows by multiplicativity of the Study determinant. And in the fifth line we used (9) again. \square

4. QUATERNION DETERMINANTAL FIELDS

Proposition 4.1. *Let $K_N(x, y) = \sum_{k=1}^N u_k(x) u_k^*(y)$, where $u_k(x)$ are orthonormal quaternionic functions, and assume that $K_N(x, y)$ is positive, that is,*

$$\text{Det}_M (K_N(x_i, x_j))|_{1 \leq i, j \leq m} \geq 0$$

for all $m \in \mathbb{Z}^+$ and all x_1, \dots, x_m . Then $K_N(x, y)$ defines a valid quaternion determinantal field.

Note that for the case when $u_k(x)$ are real quaternion function the assumption of positivity can be dropped. Indeed, for every collections of points x_1, \dots, x_m , the matrix

$$K = K_r(x_i, x_j)|_{i, j=1, \dots, m}$$

is positive semidefinite. (That is, for every real quaternion vector v , $v^* K v \geq 0$.) Hence, all eigenvalues of this matrix are real and non-negative. Since for self-dual matrices with real

quaternion entries the Moore determinant can be defined as the product of the eigenvalues, we conclude that the determinant of $K_N(x_i, x_j)$ is non-negative.

The proof of the proposition follows the standard lines. We include it for completeness.

Proof: We need to show that the functions defined by the rule

$$R_m(x_1, \dots, x_m) = \text{Det}_M(K_N(x_i, x_j)), \quad i, j = 1, \dots, m$$

are the correlation functions of a random field.

First, by assumption of positivity all functions $R_m(x_1, \dots, x_m)$ are non-negative.

Next, by integrating the kernel we find that

$$\begin{aligned} \int_{\mathbb{R}} K_N(x, x) dx &= N, \\ \int_{\mathbb{R}} K_N(x, y) K_N(y, x) dy dx &= N. \end{aligned}$$

By using formulas (1), we can conclude that the total number of points in the process with kernel $K_N(x, y)$ is exactly N . (Its expectation is N and its variance is 0.) Hence, it remains to show that the functions $R_m(x_1, \dots, x_m)$ agree among themselves for all m . This can be done by the quaternion analogue of the Mehta lemma. (Compare Theorem 5.1.4 on p. 75 in Mehta [18])

Let

$$K_m := (K(x_i, x_j))_{1 \leq i, j \leq m}$$

for a self-dual quaternion kernel K .

Lemma 4.2. *Assume that K satisfies either*

$$\int_{\mathbb{R}} K(x, y) K(y, z) dy = K(x, z) \tag{10}$$

or

$$\varphi \left(\int_{\mathbb{R}} K(x, y) K(y, z) dy \right) = \varphi(K(x, z)) + E \varphi(K(x, z)) - \varphi(K(x, z)) E, \tag{11}$$

where

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{12}$$

Then

$$\int_{\mathbb{R}} \text{Det}_M(K_m) dx_m = (N - m + 1) \text{Det}_M(K_{m-1}), \tag{13}$$

where $N = \int K(x, x) d\mu(x)$.

This result is due to Dyson. (See proof of Theorem 4 in [8].)

For the kernel $K_N(x, y)$, equation (10) holds, and we find that

$$\int_{\mathbb{R}} R_m(x_1, \dots, x_m) dx_m = (N - (m - 1)) R_{m-1}(x_1, \dots, x_{m-1}),$$

which shows that all correlation functions are all in agreement.

Corollary 4.3. *Assume that*

$$K_\xi(x, y) = \sum_{k=1}^{\infty} \xi_k u_k(x) u_k^*(y), \quad (14)$$

where every ξ_k is an independent Bernoulli random variable that takes value 1 with probability λ_k , and $u_k(x)$ are orthonormal real quaternionic functions. Suppose that $\sum_{k=1}^{\infty} \lambda_k < \infty$. Then with probability 1, the function $K_\xi(x, y)$ is a kernel of a determinantal point field, \mathcal{X}_ξ . For a given realization of variables ξ_k , the total number of points in \mathcal{X}_ξ is $\sum_{k=1}^{\infty} \xi_k$.

We can prove an analogous result for a quasi-real diagonal form.

Corollary 4.4. *Assume that*

$$K_\xi(x, y) = \sum_{k=1}^N \xi_k u_k(x) u_k^*(y), \quad (15)$$

where every ξ_k is an independent Bernoulli random variable that takes value 1 with probability λ_k . Suppose that the kernel $K(x, y) = \sum_{k=1}^N u_k(x) u_k^*(y)$ is completely positive. Then with probability 1, the function $K_\xi(x, y)$ is a kernel of a determinantal point field, \mathcal{X}_ξ . For a given realization of variables ξ_k , the total number of points in \mathcal{X}_ξ is $\sum_{k=1}^N \xi_k$.

Proof of Theorem 1.5: By the iterated expectation formula, the correlation functions of process \mathcal{X}_ξ equal $\mathbb{E} \text{Det}_M(K_\xi(x_i, x_j))$, where the expectation is taken over randomness in ξ . Hence, it is enough to prove that

$$\mathbb{E} \text{Det}_M(K_\xi(x_i, x_j)) = \text{Det}_M(K(x_i, x_j)), \quad i, j = 1, \dots, m, \quad (16)$$

almost everywhere.

First, let

$$K_R(x, y) = \sum_{k=1}^R \lambda_k u_k(x) u_k^*(y).$$

Since we assumed the absolute convergence of the kernel, hence $\text{Det}_M(K_R(x_i, x_j))$ converges to $\text{Det}_M(K(x_i, x_j))$ almost everywhere as $R \rightarrow \infty$.

Let R -by- m matrix C be defined as

$$C_{kl} = u_k^*(x_l), \quad k = 1, \dots, R; \quad l = 1, \dots, m.$$

and let Λ be an R -by- R diagonal matrix with diagonal entries λ_i .

By Corollary 3.2,

$$\text{Det}_M(K_R(x_i, x_j)) = \sum_{\substack{I=(i_1, \dots, i_m) \\ i_1 < \dots < i_m}} \lambda_{i_1} \dots \lambda_{i_m} \text{Det}_M((C^I)^* C^I). \quad (17)$$

Next, let the random variable $\text{Det}_M(K_\xi(x_i, x_j))$ be denoted as Y and let A_R be the event that all ξ_k are zero for $k > R$. (That is, $A_R = \cap_{k>R} \{\xi_k = 0\}$). Note that

$$\mathbb{E}Y = \mathbb{E}(Y|A_R) \mathbb{P}(A_R) + \mathbb{E}(Y|A_R^c) \mathbb{P}(A_R^c). \quad (18)$$

By using independence of A_R and ξ_k for $k \leq R$, we find that

$$\mathbb{E}(Y|A_R) \mathbb{P}(A_R) = \mathbb{E} \text{Det}_M(C^* \Lambda_\xi C) \mathbb{P}(A_R),$$

where Λ_ξ denotes an R -by- R diagonal matrix with diagonal entries ξ_i .

Next, by Corollary 3.2,

$$\mathbb{E} \text{Det}_M(C^* \Lambda_\xi C) = \mathbb{E} \sum_{\substack{I=(i_1, \dots, i_m) \\ i_1 < \dots < i_m}} \xi_{i_1} \dots \xi_{i_m} \text{Det}_M((C^I)^* C^I).$$

Since the variables $\xi_{i_1}, \dots, \xi_{i_m}$ are independent and have expectation $\lambda_{i_1}, \dots, \lambda_{i_m}$, we find that $\mathbb{E} \xi_{i_1} \dots \xi_{i_m} = \lambda_{i_1} \dots \lambda_{i_m}$. Hence

$$\mathbb{E}(Y|A_R) \mathbb{P}(A_R) = \text{Det}_M(K_R(x_i, x_j)) \mathbb{P}(A_R), \quad (19)$$

and the probability $\mathbb{P}(A_R)$ converges to 1 as $R \rightarrow \infty$ by independence of ξ_k , Borel-Cantelli lemma and the assumption $\sum \lambda_k < \infty$.

Now let us show that $\mathbb{E}(Y|A_R^c) \mathbb{P}(A_R^c)$ converges to zero almost everywhere. By positivity of the determinant, it is enough to show that

$$\int_{\mathbb{R}^m} \mathbb{E}(\text{Det}_M(K_\xi(x_i, x_j)) | A_R^c) \mathbb{P}(A_R^c) \rightarrow 0 \quad (20)$$

as $R \rightarrow \infty$.

Let

$$n_\xi := \sum_{k=1}^{\infty} \xi_k,$$

which is finite since both the expectation and the variance of the sum on the right hand side are convergent. Since K_ξ is a projection operator, the total number of points of the process \mathcal{X}_ξ in \mathbb{R} equals n_ξ . By changing the order of the expectation and the integral signs in (20), which is possible since the integrand is positive, and by using the identities for correlation functions we obtain that we need to estimate

$$\mathbb{E}(n_\xi(n_\xi - 1) \dots (n_\xi - m + 1) | A_R^c) \mathbb{P}(A_R^c),$$

which is smaller than

$$\mathbb{E}(n_\xi^m | A_R^c) \mathbb{P}(A_R^c).$$

By expanding

$$n_\xi^m = \left(\sum_{k=1}^{\infty} \xi_k \right)^m,$$

and using the fact that $\xi_k^s = \xi_k$ for every integer $s \geq 1$, we observe that it is enough to show that

$$\mathbb{E} \left(\sum_{i_1 < \dots < i_r} \xi_{i_1} \xi_{i_2} \dots \xi_{i_r} | A_R^c \right) \mathbb{P}(A_R^c),$$

where the sum is over all possible ordered r -tuples (i_1, \dots, i_r) such that $i_1 < \dots < i_r$, and $1 \leq r \leq m$.

We can divide the sum in two parts. The first part is when $i_r \leq R$. In this case, the variables $\xi_{i_1}, \xi_{i_2}, \dots, \xi_{i_r}$ are independent from the event A_R^c , and therefore we can estimate this part of the sum as

$$\left(\sum_{i_1 < \dots < i_r} \lambda_{i_1} \lambda_{i_2} \dots \lambda_{i_r} | A_R^c \right) \mathbb{P}(A_R^c) \leq c S^m \mathbb{P}(A_R^c),$$

where $S := \sum_{k=1}^{\infty} \lambda_k$ and c is an absolute constant. Hence this part converges to zero as $R \rightarrow \infty$ because $\mathbb{P}(A_R^c)$ converges to zero.

The second part of the sum is when $i_r > R$. In this case, the event $\xi_{i_r} = 1$ implies A_R^c and therefore,

$$\begin{aligned} \mathbb{E}(\xi_{i_1} \xi_{i_2} \dots \xi_{i_r} | A_R^c) \mathbb{P}(A_R^c) &= \mathbb{P}(\xi_{i_1} = 1, \dots, \xi_{i_r} = 1, \text{ and } A_R^c) \\ &= \mathbb{P}(\xi_{i_1} = 1, \dots, \xi_{i_r} = 1) \\ &= \lambda_{i_1} \dots \lambda_{i_r}. \end{aligned}$$

Therefore, we estimate the second part as

$$\begin{aligned} \sum_{\substack{i_1 < \dots < i_r, \\ i_r > R}} \lambda_{i_1} \dots \lambda_{i_r} &\leq \sum_{i_r=R+1}^{\infty} \sum_{i_1, \dots, i_{r-1}=1}^{\infty} \lambda_{i_1} \dots \lambda_{i_r} \\ &= S^{m-1} \sum_{i_r=R+1}^{\infty} \lambda_{i_r}. \end{aligned}$$

Hence the second part converges to zero as $R \rightarrow \infty$, because the tail sums $\sum_{i_r=R+1}^{\infty} \lambda_{i_r}$ converge to zero.

This shows that $\mathbb{E}(Y | A_R^c) \mathbb{P}(A_R^c)$ in (18) converges to zero as $R \rightarrow \infty$. If we compare (17) and (19) and let R grow to infinity, we find that

$$\mathbb{E} \text{Det}_M(K_{\xi}(x_i, x_j)) = \text{Det}_M(K(x_i, x_j)) \quad (21)$$

almost everywhere, and this completes the proof of the theorem. \square .

Proof of Theorem 1.6: The proof is essentially the same as for Theorem 1.5 but it is easier, since we assumed that the number of eigenvalues is finite. The only difference in the proof is that the matrix C is defined differently: C is s -by- m matrix defined by

$$C_{kl} = \sum_{i=1}^r v_{ki} u_i^*(x_l), \quad k = 1, \dots, s; \quad l = 1, \dots, m.$$

\square

Proof of Theorem 1.3:

This is an immediate consequence of Theorem 1.5. \square

5. A CLT FOR QUATERNION DETERMINANTAL FIELDS

Proof of Theorem 1.7: The assumptions ensure that every field \mathcal{X}_n has a kernel $K_n(x, y)$ which can be written as $\sum_{k=1}^{\infty} \lambda_k^{(n)} u_k^{(n)}(x) u_k^{(n)*}(y)$ with functions $u_k^{(n)}(x)$ which are orthonormal on D and are real quaternionic, and with real $\lambda_k^{(n)} \in [0, 1]$ that satisfy the condition $\sum_{k=1}^{\infty} \lambda_k^{(n)} < \infty$. It is clear that Theorem 1.5 holds also for quaternion determinantal fields defined on D , and therefore we can write

$$\mathcal{N}_n = \sum_{k=1}^{\infty} \xi_k^{(n)},$$

where $\xi_k^{(n)}$ are independent Bernoulli random variables, and $\mathbb{P}(\xi_k^{(n)} = 1) = \lambda_k^{(n)}$. We can truncate these infinite sums:

$$\tilde{\mathcal{N}}_n := \sum_{k=1}^{K_n} \xi_k^{(n)},$$

in such a way that $\mathcal{N}_n - \tilde{\mathcal{N}}_n$ approaches 0 in probability, $\mathbb{E}\mathcal{N}_n \rightarrow \mathbb{E}\tilde{\mathcal{N}}_n$, $\mathbb{V}ar\mathcal{N}_n \rightarrow \mathbb{V}ar\tilde{\mathcal{N}}_n$. By a suitable modification of the Lindenberg-Feller theorem ([23], Theorem III.4.2 on p. 334) we have that

$$\frac{\tilde{\mathcal{N}}_n - \mathbb{E}(\tilde{\mathcal{N}}_n)}{\sqrt{\mathbb{V}ar(\tilde{\mathcal{N}}_n)}}$$

approaches the standard Gaussian random variable and this implies the claim of the theorem.

□

The proof of Theorem 1.8 is similar.

6. AN EXAMPLE

Recall that the circular unitary ensemble (CUE) is defined as the probability space of all N -by- N unitary matrices with the Haar measure. The circular orthogonal ensemble (COE) is the space of the N -by- N symmetric unitary matrices S with the probability measure that satisfies the following invariance condition:

$$(d_H S) = (d_H V^T S V)$$

for arbitrary unitary V .

The circular symplectic ensemble (CSE) is the set of all N -by- N self-dual unitary matrices S with real quaternionic entries and with the probability measure that satisfies the following invariance condition:

$$(d_H \tilde{S}) = (d_H J_{2N}^{-1} V^T J_{2N} \tilde{S} V),$$

where $\tilde{S} = \varphi(S)$ is the $2N$ -by- $2N$ complex matrix representing S , V is an arbitrary unitary $2N$ -by- $2N$ matrix, and J_{2N} is as defined above (a block-diagonal matrix with 2-by-2 blocks $\varphi(\mathbf{j})$ on the main diagonal). For more details see Dyson's papers or Chapter 2 in Forrester's book [10].

The eigenvalues of the matrices from these ensembles are located on the unit circle and can be identified with angles θ_k on the unit circle. The density of the eigenvalue distribution for these ensembles is given by

$$C_{\beta N}^{-1} \prod_{1 \leq j < k \leq N} \left| e^{i\theta_j} - e^{i\theta_k} \right|^\beta, \quad -\pi \leq \theta_l < \pi,$$

with $\beta = 1, 2$, and 4 for COE, CUE, and CSE, respectively. (See Proposition 2.5 in [10]).

The eigenvalues form random point fields on the unit circle, which we denote \mathcal{X}_N^{COE} , \mathcal{X}_N^{CUE} , and \mathcal{X}_N^{CSE} . For $\beta = 2$, there is a well known determinantal formula for the correlation functions of the random point field, and therefore \mathcal{X}_N^{CUE} is determinantal. For $\beta = 1$ and 4 , Dyson discovered formulas for the correlation functions in terms of quaternion determinants. Therefore, fields \mathcal{X}_N^{COE} and \mathcal{X}_N^{CSE} are quaternion determinantal, and we can seek to apply the results which we found above.

In this paper we will be concerned only with the symplectic ensemble.

Let N be any positive integer. Define

$$s_{2N}(\theta) := \frac{1}{2\pi} \sum_p e^{ip\theta} = \frac{1}{\pi} \sum_{p>0} \cos(p\theta) = \frac{1}{2\pi} \frac{\sin(N\theta/2)}{\sin(\theta/2)}.$$

(Here the first summation is over $(-2N+1)/2, (-2N+3)/2, \dots, (2N-1)/2$, and the second summation is over $1/2, \dots, (2N-1)/2$.) Note that $s_{2N}(\theta)$ is even in θ .

We write

$$Ds_{2N}(\theta) := \frac{d}{d\theta} s_{2N}(\theta) = \frac{i}{2\pi} \sum_p p e^{ip\theta} = -\frac{1}{\pi} \sum_{p>0} p \sin(p\theta),$$

and

$$Is_{2N}(\theta) := \int_0^\theta s_{2N}(\theta') d\theta',$$

so that

$$Is_{2N}(\theta) = \frac{1}{2\pi i} \sum_p p^{-1} e^{ip\theta} = \frac{1}{\pi} \sum_{p>0} \frac{1}{p} \sin(p\theta)$$

The functions Ds_{2N} , and Is_{2N} are odd in θ .

Define the quaternion function $\sigma_{N4}(\theta)$ by its matrix representation:

$$\sigma_{N4}(\theta) = \frac{1}{2} \begin{pmatrix} s_{2N}(\theta) & Ds_{2N}(\theta) \\ Is_{2N}(\theta) & s_{2N}(\theta) \end{pmatrix} = \frac{1}{2\pi} \sum_{p>0} \begin{pmatrix} \cos(p\theta) & -p \sin(p\theta) \\ p^{-1} \sin(p\theta) & \cos(p\theta) \end{pmatrix}.$$

In terms of quaternions and slightly abusing notation, the kernel can be written as follows:

$$\begin{aligned} \sigma_{N4}(\theta) &= \frac{1}{2} \left(s_{2N} - \frac{1}{2} (Is_{2N} + Ds_{2N}) i + \frac{1}{2} (Is_{2N} - Ds_{2N}) j \right) \\ &= \frac{1}{2\pi} \sum_{p=1/2}^{N-1/2} (\cos p\theta + a_p \sin p\theta), \end{aligned}$$

where

$$a_p = \frac{1}{2p} [(p^2 - 1) i\mathbf{i} + (p^2 + 1) \mathbf{j}].$$

It is easy to check that $a_p^2 = -1$.

Dyson proved the following result (Compare Theorem 3 in [8]): The random field \mathcal{X}_N^{CSE} is quaternion determinantal with the kernel $\sigma_{N4}(\theta - \theta')$.

The kernel σ_{N4} is a projection on a subspace of a finite-dimensional linear space L , where

$$\begin{aligned} L &= \text{span} \left\{ e^{ip\theta}, p = \frac{-2N+1}{2}, \frac{-2N+3}{2}, \dots, \frac{2N-1}{2} \right\} \\ &= \text{span} \left\{ \frac{1}{\sqrt{\pi}} \cos p\theta, \frac{1}{\sqrt{\pi}} \sin p\theta, p = \frac{1}{2}, \frac{3}{2}, \dots, \frac{2N-1}{2} \right\}. \end{aligned}$$

Note that $\dim_{\mathbb{Q}} L = 2N$.

If we calculate the action of the kernel $\sigma_{N4}(\theta)$ in the basis $\pi^{-1/2} \cos p\theta, \pi^{-1/2} \sin p\theta$, then we find the following matrix representation of the operator with kernel σ_{N4} :

$$K = \frac{1}{2} \left\{ \begin{pmatrix} 1 & a_{1/2} \\ -a_{1/2} & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & a_{3/2} \\ -a_{3/2} & 1 \end{pmatrix} \oplus \dots \oplus \begin{pmatrix} 1 & a_{N-1/2} \\ -a_{N-1/2} & 1 \end{pmatrix} \right\}. \quad (22)$$

That is, K is a $2N$ -by- $2N$ block-diagonal matrix that has blocks

$$\frac{1}{2} \begin{pmatrix} 1 & a_p \\ -a_p & 1 \end{pmatrix}$$

on its main diagonal.

It follows that it can be written as

$$K = \sum_p v_p v_p^*,$$

where

$$v_p^* = \frac{1}{\sqrt{2}} (0, \dots, 0, 1, a_p, 0, \dots, 0),$$

and the entries 1 and a_p are at places $2p$ and $2p + 1$, respectively.

Hence the kernel $\sigma_{N4}(\theta)$ has a quasi-real diagonal form with eigenvalues $\lambda_p = 1$. It is easy to check that all N -by- N minors of the matrix K in (22) have the determinant equal to either 0 or 1, which implies that the kernel is completely positive.

It follows from Theorem 1.4 that kernel $\sigma_{N4}(\theta)$ defines a valid determinantal field. Of course, we already know this from Dyson's result. However, it is noteworthy that we proved our result by a different method, without relating the kernel to the eigenvalue distribution of a random matrix ensemble.

Clearly the restriction of this field to an interval $I = (a, b)$ is also quaternion determinantal with the kernel $\sigma_{IN4} = \mathbf{1}_I(\theta) \sigma_{N4}(\theta - \theta') \mathbf{1}_I(\theta')$. Numerical evaluations suggest that this restricted kernel is also quasi-real with positive eigenvalues for arbitrary N and I . However, the proof of this claim is elusive. The author proved it only for $N = 2$.

APPENDIX A. PROOF OF THEOREM 3.1

Since the identity is algebraic, it is enough to show that it holds for matrices with real quaternion entries. We will prove this by showing that the corresponding result holds if we write the quaternion determinants in terms of Pfaffians. Namely, let $C = X_1 + X_2\mathbf{i} + X_3\mathbf{j} + X_4\mathbf{k}$ and define complex matrices $A = X_1 + X_2i$ and $B = X_3 + X_4i$. Then, by using (8) we obtain that

$$\text{Det}_M(C^*C) = -\text{Pf} \begin{pmatrix} -B^*A + (B^*A)^t & (A^*A)^t + B^*B \\ -A^*A - (B^*B)^t & A^*B - (A^*B)^t \end{pmatrix}.$$

The blocks $-B^*A + (B^*A)^t$ and $A^*B - (A^*B)^t$ are antisymmetric, and $((A^*A)^t + B^*B)^t = -A^*A - (B^*B)^t$, so the block matrix is antisymmetric as well.

What we need to prove is that

$$\text{Pf} \begin{pmatrix} -B^*A + (B^*A)^t & (A^*A)^t + B^*B \\ -A^*A - (B^*B)^t & A^*B - (A^*B)^t \end{pmatrix} = \sum_I \text{Pf} \begin{pmatrix} -B^{I*}A^I + (B^{I*}A^I)^t & (A^{I*}A^I)^t + B^{I*}B^I \\ -A^{I*}A^I - (B^{I*}B^I)^t & A^{I*}B^I - (A^{I*}B^I)^t \end{pmatrix},$$

where the summation is over all ordered m -tuples $I = (i_1 < \dots < i_m)$, with $i_k \in \{1, \dots, n\}$, and A^I, B^I are the matrices that are obtained from matrices A and B , respectively, by taking the rows with indices in I .

In order to prove this, we recall that if R is a $2m$ -by- $2m$ antisymmetric matrix, then the pfaffian of R is defined as follows:

$$\text{Pf}(R) = \frac{1}{2^m m!} \sum_{\sigma \in S_{2m}} \text{sgn}(\sigma) \prod_{i=1}^m R_{\sigma(2i-1)\sigma(2i)}. \quad (23)$$

Next, we note that if

$$R = \begin{pmatrix} -B^*A + (B^*A)^t & (A^*A)^t + B^*B \\ -A^*A - (B^*B)^t & A^*B - (A^*B)^t \end{pmatrix},$$

then there is a formula for R_{ij} in terms of elements of A and B . This formula depends on whether i and j are greater or less than m . For example if i and j are both $\leq m$, then

$$R_{ij} = \sum_{a=1}^n (-\overline{B}_{a,i} A_{a,j} + \overline{B}_{a,j} A_{a,i}) \equiv \sum_{a=1}^n \Psi_a(i, j).$$

If we substitute this in formula (23), and expand, then we get

$$\text{Pf}(R) = \frac{1}{2^m m!} \sum_{\sigma \in S_{2m}} \text{sgn}(\sigma) \sum_{(a,b,\dots,z)} \Psi_a(\sigma(1), \sigma(2)) \dots \Psi_z(\sigma(2m-1), \sigma(2m)), \quad (24)$$

where the summation is over all m -tuples (a, b, \dots, z) with each letter taking a value in $\{1, \dots, n\}$.

A similar formula holds for the pfaffian of R^I , where

$$R^I = \begin{pmatrix} -B^{I*}A^I + (B^{I*}A^I)^t & (A^{I*}A^I)^t + B^{I*}B^I \\ -A^{I*}A^I - (B^{I*}B^I)^t & A^{I*}B^I - (A^{I*}B^I)^t \end{pmatrix}.$$

Namely,

$$\text{Pf}(R^I) = \frac{1}{2^m m!} \sum_{\sigma \in S_{2m}} \text{sgn}(\sigma) \sum_{(a,b,\dots,z) \in I^m} \Psi_a(\sigma(1), \sigma(2)) \dots \Psi_z(\sigma(2m-1), \sigma(2m)). \quad (25)$$

The difference with the previous formula is that the elements of m -tuples (a, b, \dots, z) are now restricted to take values among indices in m -tuple I .

Let us for shortness write $\Psi_{a,\dots,z}(\sigma)$ for the product $\Psi_a(\sigma(1), \sigma(2)) \dots \Psi_z(\sigma(2m-1), \sigma(2m))$.

If all elements of (a, b, \dots, z) are different, then the term $\Psi_{a,\dots,z}(\sigma)$ occurs once in expansion (24) and once in the sum of expansions (25),

$$\sum_I \text{Pf}(R^I).$$

(In this sum, it occurs in the expansion of that $\text{Pf}(R^I)$, for which I is the ordered version of the m -tuple (a, b, \dots, z) .)

If some of the elements of (a, b, \dots, z) coincide, then the situation is different. The term $\Psi_{a,\dots,z}(\sigma)$ occurs once in expansion (24) but it can occur more than once in the sum

$$\sum_I \text{Pf}(R^I).$$

For example, if all elements of the m -tuples are the same, $a = b = \dots = z$, then this term will appear in the expansion of each $\text{Pf}(R^I)$, whose index I contain a .

Clearly, in order to prove that $\text{Pf}(R) = \sum_I \text{Pf}(R^I)$, it is enough to prove that the sum of all these terms is zero. That is, it is enough to show that for a fixed m -tuple (a, b, \dots, z) with at least two elements that are equal, the sum

$$\sum_{\sigma \in S_{2m}} \text{sgn}(\sigma) \Psi_{a,\dots,z}(\sigma)$$

is zero.

Without loss of generality we can assume that $a = b$. We have to consider several cases of σ , which are summarized in the following table

	$\sigma(1)$	$\sigma(2)$	$\sigma(3)$	$\sigma(4)$
$S_{2m}[1]$				
$S_{2m}[2]$			*	*
$S_{2m}[3]$		*		*
$S_{2m}[4]$		*	*	
$S_{2m}[5]$				*
$S_{2m}[6]$		*		
$S_{2m}[7]$			*	
$S_{2m}[8]$		*	*	*

The star means that the corresponding $\sigma(i)$ is greater than m . For example, $S_{2m}[1]$ denote the set of all permutations from S_{2m} that satisfy the condition that all of $\sigma(1), \sigma(2), \sigma(3), \sigma(4)$

are smaller than or equal to m . For permutations in this set,

$$\Psi_a(\sigma(1), \sigma(2)) \Psi_a(\sigma(3), \sigma(4)) = (-\overline{B}_{a,i} A_{a,j} + \overline{B}_{a,j} A_{a,i}) (-\overline{B}_{a,k} A_{a,l} + \overline{B}_{a,l} A_{a,k}).$$

$S_{2m}[2]$ denote the set of all permutations from S_{2m} that satisfy the condition that $\sigma(1), \sigma(2)$ are smaller than or equal to m and $\sigma(3), \sigma(4)$ are greater than m , and so on.

Let us define $\tau_1[\sigma]$, as a permutation that coincides with σ on all indices except 2 and 4, for which it is defined by equalities $\tau_1[\sigma](2) = \sigma(4)$, and $\tau_1[\sigma](4) = \sigma(2)$. Similarly, $\tau_2[\sigma]$ is defined as a permutation which acts on everything except 2 and 3 as σ , and on these indices it is defined by $\tau_2[\sigma](2) = \sigma(3)$, $\tau_2[\sigma](3) = \sigma(2)$. Finally we define $\tau_3[\sigma]$ as a permutation that coincides with σ on all indices except 2, 3, and 4, where it is defined by the rules: $\tau_3[\sigma](2) = \sigma(4)$, $\tau_3[\sigma](3) = \sigma(2)$, $\tau_3[\sigma](4) = \sigma(3)$. Note that $\text{sgn}(\tau_1[\sigma]) = \text{sgn}(\tau_2[\sigma]) = -\text{sgn}(\sigma)$, and $\text{sgn}(\tau_3[\sigma]) = \text{sgn}(\sigma)$. Observe that for an arbitrary function f ,

$$\sum_{\sigma \in S_{2m}[1]} (f(\sigma) + f(\tau_1[\sigma]) + f(\tau_2[\sigma])) = 3 \sum_{\sigma \in S_{2m}[1]} f(\sigma).$$

It is easy to check that the identity

$$\begin{aligned} 0 &= \Psi_a(\sigma(1), \sigma(2)) \Psi_a(\sigma(3), \sigma(4)) - \Psi_a(\tau_1[\sigma](1), \tau_1[\sigma](2)) \Psi_a(\tau_1[\sigma](3), \tau_1[\sigma](4)) \\ &\quad - \Psi_a(\tau_2[\sigma](1), \tau_2[\sigma](2)) \Psi_a(\tau_2[\sigma](3), \tau_2[\sigma](4)) \end{aligned}$$

holds for permutations in $S_{2m}[1]$, and this implies that

$$\sum_{\sigma \in S_{2m}[1]} \text{sgn}(\sigma) \Psi_{a,\dots,z}(\sigma) = 0. \quad (26)$$

Next,

$$\sum_{\sigma \in S_{2m}[2]} (f(\sigma) + f(\tau_1[\sigma]) + f(\tau_3[\sigma])) = \sum_{\sigma \in S_{2m}[2]} f(\sigma) + 2 \sum_{\sigma \in S_{2m}[3]} f(\sigma).$$

and it is easy to check the identity

$$\begin{aligned} 0 &= \Psi_a(\sigma(1), \sigma(2)) \Psi_a(\sigma(3), \sigma(4)) - \Psi_a(\tau_1[\sigma](1), \tau_1[\sigma](2)) \Psi_a(\tau_1[\sigma](3), \tau_1[\sigma](4)) \\ &\quad + \Psi_a(\tau_3[\sigma](1), \tau_3[\sigma](2)) \Psi_a(\tau_3[\sigma](3), \tau_3[\sigma](4)). \end{aligned} \quad (27)$$

for $\sigma \in S_{2m}[2]$. Hence, identity (27) implies that

$$\left(\sum_{\sigma \in S_{2m}[2]} + 2 \sum_{\sigma \in S_{2m}[3]} \right) \text{sgn}(\sigma) \Psi_{a,\dots,z}(\sigma) = 0 \quad (28)$$

The other cases are similar. We use

$$\sum_{\sigma \in S_{2m}[4]} (f(\sigma) + f(\tau_1[\sigma]) + f(\tau_3[\sigma])) = 2 \sum_{\sigma \in S_{2m}[4]} f(\sigma) + \sum_{\sigma \in S_{2m}[2]} f(\sigma)$$

in order to conclude that

$$\left(2 \sum_{\sigma \in S_{2m}[4]} + \sum_{\sigma \in S_{2m}[2]} \right) \operatorname{sgn}(\sigma) \Psi_{a,\dots,z}(\sigma) = 0 \quad (29)$$

By adding (28) and (29), we obtain

$$\left(\sum_{\sigma \in S_{2m}[2]} + \sum_{\sigma \in S_{2m}[3]} + \sum_{\sigma \in S_{2m}[4]} \right) \operatorname{sgn}(\sigma) \Psi_{a,\dots,z}(\sigma) = 0. \quad (30)$$

Next, the identity

$$\sum_{\sigma \in S_{2m}[5]} (f(\sigma) + f(\tau_1[\sigma]) + f(\tau_3[\sigma])) = 2 \sum_{\sigma \in S_{2m}[5]} f(\sigma) + \sum_{\sigma \in S_{2m}[6]} f(\sigma)$$

implies that

$$\left(2 \sum_{\sigma \in S_{2m}[5]} + \sum_{\sigma \in S_{2m}[6]} \right) \operatorname{sgn}(\sigma) \Psi_{a,\dots,z}(\sigma) = 0, \quad (31)$$

and the identity

$$\sum_{\sigma \in S_{2m}[6]} (f(\sigma) + f(\tau_1[\sigma]) + f(\tau_3[\sigma])) = \sum_{\sigma \in S_{2m}[6]} f(\sigma) + 2 \sum_{\sigma \in S_{2m}[7]} f(\sigma)$$

implies that

$$\left(\sum_{\sigma \in S_{2m}[6]} + 2 \sum_{\sigma \in S_{2m}[7]} \right) \operatorname{sgn}(\sigma) \Psi_{a,\dots,z}(\sigma) = 0. \quad (32)$$

By adding (31) and (32), we obtain:

$$\left(\sum_{\sigma \in S_{2m}[5]} + \sum_{\sigma \in S_{2m}[6]} + \sum_{\sigma \in S_{2m}[7]} \right) \operatorname{sgn}(\sigma) \Psi_{a,\dots,z}(\sigma) = 0. \quad (33)$$

Finally,

$$\sum_{\sigma \in S_{2m}[8]} (f(\sigma) + f(\tau_1[\sigma]) + f(\tau_3[\sigma])) = 3 \sum_{\sigma \in S_{2m}[8]} f(\sigma).$$

Identity (27) still holds in this case, and therefore,

$$\sum_{\sigma \in S_{2m}[8]} \operatorname{sgn}(\sigma) \Psi_{a,\dots,z}(\sigma) = 0. \quad (34)$$

Clearly, the sets $S_{2m}[k]$, are disjoint and their union over $k = 1, \dots, 8$ consists of all permutations from S_{2m} for which $\sigma(1) \leq m$. Hence, identities (26), (30), (33), and (34) imply that

$$\sum_{\substack{\sigma \in S_{2m} \\ \sigma(1) \leq m}} \operatorname{sgn}(\sigma) \Psi_{a,\dots,z}(\sigma) = 0.$$

The case $\sigma(1) > m$ can be handled similarly, and we obtain

$$\sum_{\sigma \in S_{2m}} \operatorname{sgn}(\sigma) \Psi_{a,\dots,z}(\sigma) = 0.$$

This holds provided the first two indices in (a, \dots, z) are equal. It is clear that this identity also holds if any two indices in (a, \dots, z) are equal.

As observed earlier, this implies that we can remove all terms $\Psi_{a, \dots, z}(\sigma)$, for which two indices in (a, \dots, z) coincide, from the expansions of both $\text{Pf}(R)$ and the sum $\sum_I \text{Pf}(R^I)$. (See expansions (24) and (25).) Then it is clear that the remaining terms in the expansions are the same. Hence

$$\text{Pf}(R) = \sum_I \text{Pf}(R^I),$$

and this implies the statement of the theorem. \square

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